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Positive Elementary Solutions and Completely Monotonic Functions

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1. INTRODUCTION

The role of elementary solutions in the theory of linear partial differential equations is well known. Since many properties of the solutions of various particular problems are mirrored by properties of the corresponding Green's functions, which are elementary solutions adapted to a particular problem, it is possible to determine general properties of such solutions by studying corresponding properties of suitable elementary solutions. This paper is devoted to a study of positivity of solutions of a class of parabolic and hyperbolic linear partial differential equations that arise as equations of motion of viscoelastic media. It will be shown that the elementary solutions in R^n of these equations have a positivity property. This result generalizes a well-known property of the heat flow operator in n dimensions, and also certain results of Weinberger for the wave equation with constant coefficients in n dimensions.

The method we adopt is to use Laplace transforms and the properties of completely monotonic functions [19]. These are outlined in Sections 3 and 6 below. A preliminary account of the method has been given in [7], however more complete results in higher dimensions, and extensions to certain anisotropic equations, are now available.

Positivity for solutions of a linear differential equation is frequently related to a maximum principle. We conclude this account by showing that a maximum principle can be utilized but for a slightly different class of equations for which at present no physical interpretation is known. This latter method also adapts to certain boundary conditions of the third kind.

2. Some Classical Examples

Let $x = \{x_1, x_2, \ldots, x_n\}$ denote Cartesian coordinates in R^n and let

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{2.1}$$

be the Laplacian operator in R^n. The heat flow equation in R^n with source point at the origin is

$$\frac{\partial u}{\partial t} - \Delta u = \delta(x)\,\delta(t), \tag{2.2}$$

where $u = u(x, t)$ is an elementary solution, assumed to vanish identically in x for $t < 0$. Also $\delta(x)$, $\delta(t)$ denote the Dirac distribution in R^n and R, respectively. In this case the elementary solution is well known to be the nonnegative function

$$u(x, t) = \frac{H(t)}{2^n(\pi t)^{n/2}} \exp\left(-\frac{r^2}{4t}\right),$$

where

$$r^2 = \sum_{i=1}^{n} x_i^2$$

and the Heaviside function $H(t)$ denotes zero for $t < 0$, unity for $t \geqslant 0$.

For the classical wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = \delta(x)\,\delta(t) \tag{2.3}$$

with $u = 0$ for $t < 0$, the elementary solution in R^n is

$$\delta^{(n-3)/2}(t^2 - r^2)\,H(t), \tag{2.4}$$

and the positivity of the elementary solution itself depends on the dimension n of the space variables. For $n = 1$, 2 and 3, the elementary solution is positive, as can be seen from the above expression or verified from the well-known particular forms in these cases. However, for $n \geqslant 4$ the elementary solution itself is not positive. Nonetheless, as the above expression shows, the elementary solution is a derivative of order $(n - 3)/2$ of a positive distribution. (For the definition of a positive distribution see for instance [10, p. 142]; essentially T is positive if $T(\varphi) \geqslant 0$ for all $\varphi \geqslant 0$). This remark is related to a result of Weinberger [18]; it will be shown below that a similar result holds for all viscoelastic wave equations of which the classical wave equation is one of the primary cases.

3. Laplace Transforms and Completely Monotonic Functions

Elementary solutions such as those of the foregoing examples are often evaluated by means of their Laplace transforms. We define the Laplace transform $\tilde{u}(s)$ of a given function $u(t)$ $(0 \leqslant t < \infty)$ as follows:

$$\tilde{u}(s) = \int_0^{\infty} e^{-st} u(t)\, dt. \tag{3.1}$$

If the determining function $u(t)$ is nonnegative it is easily verified that on the positive s-axis (wherever convergence holds)

$$(-1)^k\, \tilde{u}^{(k)}(s) = \int_0^{\infty} s^k e^{-st} u(t)\, dt \geqslant 0 \qquad 0 \leqslant s < \infty.$$

We shall use the basic theorem on the representation of Laplace transforms, which is due to Bernstein [19, p. 160] and asserts that this sequence of conditions is necessary and sufficient in order that $u(t) \geqslant 0$.

A function f defined on a real interval $a \leqslant s \leqslant b$, is said to be *completely monotonic* on that interval if

$$(-1)^k f^{(k)}(s) \geqslant 0 \qquad a \leqslant s \leqslant b. \tag{3.2}$$

Hence we have the theorem of Bernstein: In order that $f : s \to f(s)$ should be completely monotonic on $0 \leqslant s < \infty$, it is necessary and sufficient that f should be representable as a Laplace transform

$$f(s) = \int_0^{\infty} e^{-st}\, d\alpha(t) \tag{3.3}$$

where $\alpha(t)$ is nondecreasing for $0 \leqslant t < \infty$, and where the integral converges for $0 < s < \infty$.

The class of completely monotonic functions on $(0, \infty)$ forms a cone having the following inclusion properties: if $f_1(s)$ and $f_2(s)$ are completely monotonic then

$$c_1 f_1(s) + c_2 f_2(s) \qquad c_1 \geqslant 0, c_2 \geqslant 0$$

$$(-1)^k f_1^{(k)}(s) \qquad \text{and} \qquad f_1(s) f_2(s)$$

are also completely monotonic.

In general the composed function $f_1(f_2(s))$ is not completely monotonic. However, we shall require the following Lemma which in effect shows a property of composition of indefinite integrals of completely monotonic functions.

LEMMA 1. *If $f(s)$ and $g'(s)$ are completely monotonic and if the range of g belongs to the domain of f then $F(s) = f(g(s))$ is completely monotonic.*

PROOF. We proceed by differentiation; and induction on the order of derivatives

$$
\begin{aligned}
F(s) &= f(g(s)) \geqslant 0\\
F'(s) &= f'(g(s))\, g'(s) \leqslant 0\\
F''(s) &= f''(g(s))\, (g'(s))^2 + f'(g(s))\, g''(s) \geqslant 0\\
F'''(s) &= f'''(g(s))\, (g'(s))^3 + 3f''(g(s))\, g'(s)\, g''(s) + f'(g(s))\, g'''(s) \leqslant 0.\\
&\ \vdots
\end{aligned}
$$

Assuming that $(-1)^k F^{(k)}(s) \geqslant 0$ for $k \leqslant n$, we observe that the terms to be differentiated at the $n+1$st stage are products of derivatives of f and of g', and that one factor in each such product will be differentiated in each term of the required expression for $F^{(n+1)}(s)$. If a factor $f^{(h)}(g(s))$ is differentiated, a result $f^{(h+1)}(g(s))\, g'(s)$ is obtained which has the opposite sign since $g'(s) > 0$. Terms involving differentiation of $g^{(\ell)}(s)$ also result in a change of sign, and we conclude that the sign of $F^{(n+1)}(s)$ is always opposed to that of $F^{(n)}(s)$. The proof is now completed by induction on n.

For $\alpha > 0$, $\alpha \in R$, the particular functions

$$e^{-\alpha s}, \qquad s^{-\alpha}$$

are completely monotonic on $0 < s < \infty$. Similarly the functions

$$-e^{-\alpha s}, \qquad \frac{s^{1-\alpha}}{1-\alpha} \ (\alpha \neq 1), \qquad \log s, \qquad \log(\alpha s + \beta), \qquad \alpha, \beta > 0$$

are functions having completely monotonic first derivatives on $0 < s < \infty$.

We also employ the following lemma.

LEMMA 2. *If $0 < \beta < \alpha < \infty$ and $\gamma > 0$ then $[(s+\alpha)/(s+\beta)]^\gamma$ is completely monotonic for $0 < s < \infty$.*

PROOF. Note that

$$
\begin{aligned}
\left(\frac{s+\alpha}{s+\beta}\right)^\gamma &= \exp(-\gamma[\log(s+\beta) - \log(s+\alpha)])\\
&= \exp(-\gamma G(s)),
\end{aligned}
$$

where

$$G(s) = \log(s+\beta) - \log(s+\alpha).$$

Now

$$G'(s) = \frac{1}{s+\beta} - \frac{1}{s+\alpha} > 0$$

and

$$(-1)^k G^{(k+1)}(s) = \frac{k!}{(s+\beta)^{k+1}} - \frac{k!}{(s+\alpha)^{k+1}} > 0, \qquad 0 < \beta < \alpha.$$

Hence $G'(s)$ is completely monotonic. Noting that $f(s) = e^{-\gamma s}$ is completely monotonic and applying Lemma 1 with $g(s) = G(s)$, we find that the result follows at once.

4. Stokes' Equation

As a preliminary example of the use of algebraic techniques we shall now consider the Stokes' equation in one space dimension. After the heat flow and wave equations, this is the simplest of the viscoelastic types. Stokes' equation for $n = 1$ has the form

$$u_{tt} - u_{xx} - u_{xxt} = \delta(x)\,\delta(t); \tag{4.1}$$

this equation also occurs in a variety of hydrodynamical problems involving viscosity [13, 14].

Setting

$$U(s, x) = \int_0^\infty e^{-st} u(t, x)\,dt,$$

we find for $U(s, x)$ the ordinary differential equation

$$s^2 U - (1+s)\,U_{xx} = \delta(x).$$

The Green's function for the real line is found to be

$$U(s, x) = \frac{1}{2s\sqrt{1+s}} \exp\left(\frac{-s\,|\,x\,|}{\sqrt{1+s}}\right). \tag{4.2}$$

We set

$$g(s) = \frac{s}{\sqrt{1+s}} = \sqrt{1+s} - \frac{1}{\sqrt{1+s}}$$

and observe that $g'(s)$ is completely monotonic. With $f(s) = e^{-|x|s}$ and by Lemma 1 we see that the exponential factor is completely monotonic. The algebraic factors involving negative powers are also completely monotonic, so by the product property mentioned earlier we deduce that $U(s, x)$ is completely monotonic for $0 < s < \infty$. We conclude that the elementary solution of Stokes' equation is positive for $t > 0$. Asymptotic studies of this solution have been made in [9] and [13].

5. The General Viscoelastic Wave Equation

The scalar wave equation for a general viscoelastic medium has the form

$$P(D)\, D^2 U = Q(D)\, \Delta U, \qquad D \equiv \frac{\partial}{\partial t}, \tag{5.1}$$

where $P(s)$ and $Q(s)$ are polynomials with the following properties [2, Chap. 2], [11]:

(1) For $s > 0$, $P(s)$ and $Q(s)$ are positive.

(2) The zeros of $P(s)$ and $Q(s)$ are real, simple, and nonpositive.

(3) The zeros of $P(s)$ and $Q(s)$ separate each other, with the zero of least magnitude belonging to $Q(s)$.

As shown by Bland [2, p. 45] these properties give rise to four cases according as the leading zero of $Q(s)$ is zero or negative, and whether the most negative zero belongs to $P(s)$ or $Q(s)$. Thus

$$P(s) = \prod_{j=1}^{m} (s + \alpha_j)$$

$$Q(s) = \prod_{j=1}^{m+1} (s + \beta_j), \tag{5.2}$$

where

$$0 \leqslant \beta_1 < \alpha_1 < \beta_2 < \cdots < \alpha_m < \beta_{m+1} \,.$$

In the two cases when the degrees of P and Q are equal, the factor $s + \beta_{m+1}$ does not appear in Q.

The algebraic properties and the number and values of the roots of the polynomials $P(s)$ and $Q(s)$ are determined by the type of the viscoelastic model adopted to describe the given medium. The viscoelastic medium is envisioned as consisting of small elements each of which is composed in a certain way of the two basic elements which are the elastic element, or spring, and the viscous element, or dashpot.

To give some indication of the range of equations involved, we describe the simplest example of each of the four standard cases.

CASE I. *Viscous element:* $Q(s) = s$, $P(s) = 1$, the equation of heat flow in the form

$$V_{tt} = \Delta V_t .$$

With $U = V_t$, we obtain $U_t = \Delta U$.

CASE II. *Elastic element:* $Q(s) = 1$, $P(s) = 1$, the wave equation appears as

$$V_{tt} = \Delta V.$$

Recall that for this equation the elementary solution is not positive for $n > 3$.

CASE III. *Elastic and viscous elements in parallel* (the so-called Voigt element): $Q = s + 1$, $P = 1$, the Stokes equation applies in the form

$$V_{tt} = \Delta V + \Delta V_t .$$

CASE IV. *Elastic and viscous elements in series* (the so-called Maxwell element): $Q = s$, $P = s + 1$, a form of the damped wave equation appears, namely,

$$V_{ttt} + V_{tt} = \Delta V_t .$$

Setting $U = V_t$, we obtain $U_{tt} + U_t = \Delta U$, in turn reducible with $W = Ue^{\frac{1}{2}t}$ to $W_{tt} = \Delta W + \frac{1}{4} W$.

Returning to the general case, we define the Laplace transform of u with respect to t by

$$U(s, x) = \int_0^\infty e^{-st} u(t, x)\, dt. \tag{5.3}$$

The transformed equation for the elementary solution is found to be

$$s^2 P(s)\, U - Q(s)\, \Delta U = \delta(x). \tag{5.4}$$

For $n = 1$ the solution for the real line is

$$U(s, x) = \frac{1}{2s\sqrt{P(s)\, Q(s)}} \exp\left(- s \sqrt{\frac{P(s)}{Q(s)}}\, | x |\right), \tag{5.5}$$

while for $n > 1$ the elementary solution of this Helmholtz equation is

$$U(s, x) = C_n \frac{s^{(n-2)/2} P(s)^{(n-2)/4}}{Q(s)^{(n+2)/4}}\, r^{-(n-2)/2} K_{(n-2)/2}\left(s \sqrt{\frac{P(s)}{Q(s)}}\, r\right), \tag{5.6}$$

where

$$K_\nu(z) = \int_0^\infty \exp(- z \cosh w) \cosh \nu w \, dw \tag{5.7}$$

is the Bessel function of imaginary type and of the second kind [17, p. 181]. The integral representation (5.7) shows at once that $K_\nu(z)$ is completely monotonic.

6. Functions of Class P'

To prove that the Laplace transforms given above are completely monotonic for $0 < s < \infty$, we will demonstrate that the function

$$\phi_1(s) = s \sqrt{\frac{P(s)}{Q(s)}} \tag{6.1}$$

has a completely monotonic first derivative for $0 < s < \infty$. For this purpose we introduce functions of the complex variable s of the class P' studied by Aronszajn and Donoghue [1]. A function $\phi(s)$ of the class P' is, by definition

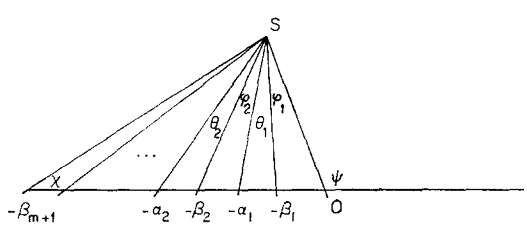
(1) holomorphic having $\operatorname{Im} \phi(s) > 0$ in the half plane $\operatorname{Im} s > 0$

(2) holomorphic and non negative on the positive real axis $s > 0$.

Lemma 3.

$$\phi_1(s) \in P'.$$

To show that $\phi_1(s) \in P'$, we first observe that condition (2) is obviously satisfied. Let $\operatorname{Im} s > 0$, and join the point s of the complex plane to the real roots of $P(s)$ and $Q(s)$, and the origin, by straight line segments. Let angles ψ, χ, θ_j, φ_j be defined as shown. Consider



$$\begin{aligned}\arg \phi_1(s) &= \arg s + \tfrac{1}{2} \sum_j \arg(s + \alpha_j) - \tfrac{1}{2} \sum_j \arg(s + \beta_j)\\ &= \arg s - \tfrac{1}{2} \sum_j \theta_j - \tfrac{1}{2} \epsilon \chi,\end{aligned}$$

where ϵ is 0 or 1 according as degree Q is equal to degree P or one unit greater, and where

$$\theta_j = \arg(s + \beta_j) - \arg(s + \alpha_j) > 0, \qquad j = 1,..., m$$
$$\chi = \arg(s + \beta_{m+1}).$$

By the external angle theorem,

$$\psi = \chi + \sum \theta_j + \sum \varphi_j ,$$

and consequently, since $\psi = \arg s$,

$$\arg \phi_1(s) = \tfrac{1}{2}\psi + \tfrac{1}{2}\sum \varphi_j + \tfrac{1}{2}(1 - \epsilon)\chi > 0.$$

Since $\operatorname{Im} s > 0$ we have $0 < \psi < \pi$, and from the sum of angles in a triangle,

$$0 < \chi + \sum \varphi_j < \pi.$$

Since $\arg \phi_1(s)$ is the average of two numbers each between 0 and π, we have $0 < \arg \phi_1(s) < \pi$ for $\operatorname{Im} s > 0$. Hence finally $\operatorname{Im} \phi_1(s) > 0$ for $\operatorname{Im} s > 0$ and thus condition (1) is satisfied. Therefore $\phi_1(s) \in P'$.

As shown by Aronszajn and Donoghue, every $\phi(s) \in P'$ has a unique integral representation of the form

$$\phi(s) = \alpha s + \beta + \int_{-\infty}^{0} \left(\frac{1}{\lambda - s} - \frac{1}{\lambda}\right) d\mu(\lambda), \tag{6.2}$$

where

(a) $\mu(\lambda)$ is a nonnegative (Borel) measure

and

(b) $(1 + \lambda^2)^{-1}$ is integrable on $(-\infty, 0]$ with respect to $\mu(\lambda)$.

LEMMA 4. *If $\Phi(s) \in P'$ then $\phi'(s)$ is completely monotonic for $0 < s < \infty$.*

PROOF. By differentiation,

$$\phi'(s) = \alpha + \int_{-\infty}^{0} \frac{d\mu(\lambda)}{(\lambda - s)^2}, \qquad s > 0,$$

where the integral is absolutely convergent in view of condition (b) above, and uniformly convergent for $0 < \delta < s < \infty$, for any $\delta > 0$.

Thus $\phi'(s) > 0$, and by further differentiations we can deduce

$$(-1)^k \phi_{(s)}^{(k+1)} = (k + 1)! \int_{-\infty}^{0} \frac{d\mu(\lambda)}{(s - \lambda)^{k+1}} \geqslant 0.$$

This completes the proof of the lemma.

Applying Lemmas 3 and 4 to $\phi_1(s)$, we conclude that $\phi_1'(s)$ is completely monotonic. Referring again to the expressions (5.5) and (5.6) for the elementary solutions, we see by Lemma 1 that the functions

$$\exp(-\phi_1(s) \mid x \mid) \qquad \text{and} \qquad K_{(n-2)/2}(\phi_1(s)\, r)$$

are each completely monotonic for $0 < s < \infty$.

7. The Positivity Theorem for Dimensions $n = 1$, 2 and 3

Since any expression of the form $s^{-\gamma}$, or $(s + \alpha)^{-\gamma}$, where $\alpha > 0$, $\gamma > 0$ determines a function completely monotonic on $0 < s < \infty$, and since the product of completely monotonic functions is again completely monotonic we see that the transformed expression for $n = 1$, namely,

$$\frac{1}{2s\sqrt{P(s)\,Q(s)}} \exp\left(-s\sqrt{\frac{P(s)}{Q(s)}} \mid x \mid\right) \tag{7.1}$$

is completely monotonic. Note also that after multiplication by s the expression is still completely monotonic.

For $n = 2$, the expression in question is

$$\frac{C_2}{Q(s)} K_0\left(s\sqrt{\frac{P(s)}{Q(s)}}\, r\right), \tag{7.2}$$

which in view of the preceding results is also completely monotonic.

For $n = 3$, we may use the formula

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

and express the transform as

$$\frac{1}{4\pi r Q(s)}\, e^{-s\sqrt{\frac{P(s)}{Q(s)}}\, r}.$$

Again it is evident that complete monotonicity holds. Summarizing we have

Theorem 1. *For $n = 1, 2$ and 3, the elementary solution vanishing at infinity of the viscoelastic wave equation*

$$P(D)\, D^2 U - Q(D)\, \Delta U = \delta(x)\, \delta(t),$$

where $D \equiv \partial/\partial t$, is positive for $t > 0$, while for $n = 1$ the first time derivative is also positive.

It will be seen below that for space dimension higher than 3, the result holds but only in a weaker form.

8. Positivity for $n \geqslant 3$

Returning to the general expression (5.6) for $n > 3$, we note that the Bessel function factor is completely monotonic. However, we shall now strengthen this result.

Lemma 5. *The function $\sqrt{z}\, K_\nu(z)$ is completely monotonic for $|\nu| \geqslant \frac{1}{2}$, $0 < z < \infty$.*

Proof. Let $y(z) = \sqrt{z}\, K_\nu(z)$, then the differential equation

$$y''(z) - \left[1 + \frac{\nu^2 - \frac{1}{4}}{z^2}\right] y = 0$$

is satisfied. Since the coefficient of y is the negative of a completely monotonic function for $|\nu| \geqslant \frac{1}{2}$, $z > 0$, it follows from [12, p. 508] (in particular the case $d = 2$ of Corollary 2.2), that this differential equation has a completely monotonic solution. This solution has the form

$$\sqrt{z}\,(C_1 K_\nu(z) + C_2 I_\nu(z));$$

but if $C_2 \neq 0$ it follows from the asymptotic behavior of $I_\nu(z)$ that this solution will be ultimately increasing. Since this would contradict the complete monotonicity property, we must have $C_2 = 0$ and the result follows.

This result can also be established by determining the Laplace-Stieltjes transform of $\sqrt{z}\, K_\nu(z)$.

We now express the Laplace transform (5.6) in the form

$$C_n \frac{s^{(n-3)/2}}{Q(s)} \left(\frac{P(s)}{Q(s)}\right)^{(n-3)/4} r^{-(n-1)/2} \left[\sqrt{rs\sqrt{\frac{P(s)}{Q(s)}}}\, K_{(n-2)/2}\left(rs\sqrt{\frac{P(s)}{Q(s)}}\right)\right]. \tag{8.1}$$

By Lemmas 1, 3, 4, and 5 the factor in the square brackets is completely monotonic for $0 < s < \infty$. By Lemma 2 the factor involving the quotient $P(s)/Q(s)$ is also completely monotonic (the presence of an extra factor in $Q(s)$

only strengthens this result). The factor $Q(s)^{-1}$ is also completely monotonic. We observe therefore, that the expression has the form

$$s^{(n-3)/2}F(s),$$

where $F(s)$ is completely monotonic.

Observe that if $F_1(s) = s^\alpha F(s)$, $\alpha > 0$ is the Laplace transform of $f_1(t)$, then $F(s) = s^{-\alpha}F_1(s)$ is the Laplace transform of the Riemann-Liouville fractional integral [15]

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau.$$

Transforming back to the time variable, we obtain the following theorem.

THEOREM 2. *For $n \geqslant 3$, the elementary solution $G(t, x)$ vanishing at infinity has the property*

$$I^\alpha G(t, x) \geqslant 0 \tag{8.2}$$

whenever $\alpha \geqslant (n-3)/2$ and $t > 0$.

Note that in the case of the wave equation the elementary solution is identically zero in certain lacunary regions. For an equation of parabolic type the strict inequality can however frequently be shown.

If $G(t, x)$ is the elementary solution for the operator

$$L = P(D)\, D^2 - Q(D)\, \Delta,$$

then $I^\alpha G(t, x)$ is an elementary solution for the operator $D^\alpha L$. By means of this observation, we can generalize to the viscoelastic class of equations a result for the wave equation due to Weinberger [18], Sather [16].

THEOREM 3. *Let $\alpha = (n-2)/2$ for n even, $\alpha = (n-3)/2$ for n* odd. *If*

$$U(0, x) = U_t(0, x) = \cdots = U_t^{(m+\alpha)}(0, x) = 0$$

and if

$$U_t^{(m+\alpha+1)}(0, x) \geqslant 0$$

and

$$D^\alpha L U(t, x) \geqslant 0, \qquad t > 0$$

then

$$U(t, x) \geqslant 0, \qquad t > 0.$$

PROOF. Suppose the homogeneous initial conditions listed above are satisfied, and that

$$U_t^{(m+\alpha+1)}(0, x) = f(x) \geqslant 0$$
$$D^\alpha LU(t, x) = F(t, x) \geqslant 0.$$

Then the solution $U(t, x)$ of this initial value problem is well-determined and is given by the representation formula

$$U(t, x) = \int_{R^n} I^\alpha G(t, x - \xi) f(\xi)\, d\xi + \int_0^t \int_{R_n} I^\alpha G(t - \tau, x - \xi) F(\tau, \xi)\, d\tau\, d\xi,$$

The terms on the right are clearly nonnegative, so the result follows.

9. Extension to Anisotropic Equations

The method of the preceding section can be extended to include equations of an anisotropic character in which different stress operators $Q_j(D)$ are present for each coordinate x_j, $j = 1,..., n$. Such equations of the form

$$P(D)\, D^2 u = \sum_{j=1}^{n} Q_j(D)\, u_{x_j x_j} + \delta(x)\, \delta(t) \tag{9.1}$$

will be treated under the assumption that for each j, $j = 1,..., n$, $P(D)$ and $Q_j(D)$ satisfy the four conditions listed at the beginning of 5. Taking the Laplace transform with respect to t, we obtain

$$P(s)\, s^2 \tilde{u} = \sum_{j=1}^{n} Q_j(s)\, \tilde{u}_{x_j x_j} + \delta(x). \tag{9.2}$$

To construct the elementary solution we define new variables

$$\xi_j = s \sqrt{\frac{P(s)}{Q_j(s)}}\, x_j = \phi_j(s)\, x_j\,, \tag{9.3}$$

where

$$\phi_j(s) = s \sqrt{\frac{P(s)}{Q_j(s)}} \qquad j = 1,..., n. \tag{9.4}$$

Then

$$\delta(x) = \phi_1(s) \cdots \phi_n(s)\, \delta(\xi),$$

while if Δ' denotes the Laplacian with respect to the ξ_j, we find

$$\Delta'\tilde{u} - \tilde{u} = -\frac{\phi_1(s)\cdots\phi_n(s)\,\delta(\xi)}{s^2P(s)}$$
$$= -\,C(s)\,\delta(\xi), \tag{9.5}$$

where

$$C(s) = \frac{s^{n-2}P(s)^{(n-2)/2}}{\prod_{j=1}^{n} Q_j(s)^{1/2}}. \tag{9.6}$$

Consequently the elementary solution has the form

$$\tilde{u}(s, x) = c_1 C(s)\,\rho^{-(n-2)/2} K_{\frac{1}{2}(n-2)}(\rho); \tag{9.7}$$

here c_1 is a positive number and $K_\nu(z)$ is given by (5.7). Here also

$$\rho^2 = \sum_{j=1}^{n} \xi_j^2 = \sum_{j=1}^{n} \phi_j^2(s)\,x_j^2 = s^2P(s)\sum_{j=1}^{n}\frac{x_j^2}{Q_j(s)}. \tag{9.8}$$

We may now write (9.7) in the form

$$\tilde{u}(s, x) = c_1 \frac{s^{n-2}P(s)^{(n-2)/2}}{\prod_{j=1}^{n} Q_j(s)^{1/2}}\,\rho^{-(n-1)/2}\sqrt{\rho}\,K_{\frac{1}{2}(n-2)}(\rho)$$

$$= c_1 \frac{s^{(n/2)-(3/2)}P(s)^{(n/4)-(3/4)}}{\prod_{j=1}^{n} Q_j(s)^{1/2}}\,\frac{1}{\left(\sum_{j=1}^{n}\frac{x_j^2}{Q_j(s)}\right)^{(n-1)/4}}\sqrt{\rho}\,K_{\frac{1}{2}(n-2)}(\rho)$$

$$= c_1 \frac{s^{(3n-7)/4}}{P(s)}\prod_{j=1}^{n}\left(\frac{P(s)}{Q_j(s)}\right)^{1/2}\left(\frac{1}{sP(s)\sum_{j=1}^{n}\frac{x_j^2}{Q_j(s)}}\right)^{(n-1)/4}\sqrt{\rho}\,K_{(n-2)/2}(\rho)$$

$$= s^{(3n-7)/4}F_1(s), \tag{9.9}$$

where $F_1(s)$ is thus defined.

To show that $F_1(s)$ is completely monotonic, we employ the following lemma.

Lemma 6. *The function*

$$\rho = \Phi(s) = s\sqrt{P(s)\sum_{j=1}^{n}\frac{x_j^2}{Q_j(s)}}. \tag{9.10}$$

is of class P'.

PROOF. Write

$$\rho = \Phi(s) = s^{1/2}\Phi_2(s)^{1/2},$$

where

$$\Phi_2(s) = sP(s) \sum_{j=1}^{n} \frac{x_j^2}{Q_j(s)}$$

$$= \sum_{j=1}^{n} \Phi_{2j}(s) \tag{9.11}$$

with

$$\Phi_{2j}(s) = \frac{sP(s)}{Q_j(s)} x_j^2$$

$$= \frac{s \prod_{k=1}^{m} (s + \alpha_k)}{\prod_{k=1}^{m+1} (s + \beta_{kj})}, \tag{9.12}$$

where $0 \leqslant \beta_1 < \alpha_1 < \beta_2 < \cdots < \alpha_m < \beta_{m+1}$, and β_{m+1} may or may not appear. In the triangle diagram on p. 476 we see that

$$\arg \frac{s}{s + \beta_1} = \phi_1; \qquad \arg \frac{s + \alpha_k}{s + \beta_{k+1}} = \phi_{k+1}, \ldots.$$

whence

$$\arg \Phi_{2j}(s) = \sum_{k=1}^{m} \phi_k + \epsilon\chi,$$

where ϵ is defined as 0 if $\deg Q = m$ and 1 if $\deg Q = m + 1$.

From the triangle in the diagram we see that

$$0 < \sum_{k=1}^{m} \phi_k + \epsilon\chi < \pi$$

whenever s lies in the upper half-plane. Since $\Phi_{2j}(s)$ is clearly real analytic and positive on the positive s-axis, we conclude that $\Phi_{2j}(s) \in P'$. By the addition property, we have $\Phi_2(s) \in P'$.

Since $s \in P'$ and $\Phi(s)$ is the geometric mean of two elements of P', we now conclude $\Phi(s) \in P'$ as stated in Lemma 6.

Returning to the expression (9.9), we see that the factors $P(s)^{-1}$ and $(P(s)/Q(s))^{1/2}$ are completely monotonic using Lemma 2. The next factor contains negative powers of $\Phi_2(s)$, which latter belongs to P', so that this factor is also completely monotonic. Finally, Lemma 6, when combined with Lemmas 1 and 5, shows that

$$\rho^{1/2} K_{(n-2)/2}(\rho)$$

is completely monotonic for $|n-2| \geqslant 1$. Excepting the cases $n = 1$ (trivial) and $n = 2$, therefore, we have the following conclusion.

THEOREM 4. *For $n \geqslant 3$, the elementary solution $G(t, x)$ vanishing at infinity of* (9.1) *has the property*

$$I^{\alpha} G(t, x) \geqslant 0 \tag{9.13}$$

whenever $\alpha \geqslant (3n-7)/4$.

Note that for $n = 3$, this gives a result weaker by the order of derivatives $\frac{1}{2}$, and by more for higher dimensions.

For $n = 2$ the expression is

$$Q_1(s)^{-\frac{1}{2}} Q_2(s)^{-\frac{1}{2}} K_0(\rho),$$

which is obviously completely monotonic so that the elementary solution itself is positive for $n = 2$.

We conclude this section with an example from hydromagnetics. It is well known that magnetohydrodynamic waves of the Alfvén type undergo one-dimensional wave propagation. However the presence of electrical resistance introduces a dissipative term of a three-dimensional character, so that a typical magnetic field component or velocity component satisfies an equation of the modified Stokes' type [5, p. 38]

$$\frac{\partial^2 h}{\partial^2 t} = \frac{\partial^2 h}{\partial z^2} + \Delta \frac{\partial h}{\partial t}. \tag{9.14}$$

We will show that the elementary solution in R^3 for $t > 0$ for this equation is also positive.

If

$$G_{tt} = G_{zz} + \Delta G_t + \delta_3(x)\,\delta(t)$$

and

$$\tilde{G}(s, x) = \int_0^{\infty} G(t, x)\, e^{-st}\, dt,$$

then

$$s^2 \tilde{G}^2 = \tilde{G}_{zz} + s\Delta\tilde{G} + \delta_3(x).$$

Set

$$\Delta' U = U_{xx} + U_{yy} + \left(1 + \frac{1}{s}\right) U_{zz}.$$

Then

$$\Delta' \tilde{G} - s\tilde{G} = -\frac{1}{\sqrt{s(s+1)}}\,\delta(x)\,\delta(y)\,\delta(z).$$

The elementary solution is

$$\tilde{G} = \frac{\exp(-\sqrt{s}\,r')}{4\pi\sqrt{s(s+1)}\,r'},$$

where

$$r'^2 = x^2 + y^2 + \frac{s}{s+1}\,z^2.$$

Note that if $\operatorname{Im} s > 0$ then $\operatorname{Im} s/(s+1) > 0$ and hence $\operatorname{Im} r' > 0$, so that $r'^2 \in P'$. Since $s \in P'$ it follows that $\sqrt{s}\,r' \in P'$, as the geometric mean of two elements of P' will again be an element of P'. Also

$$\frac{1}{\sqrt{s(s+1)}\,r'} = \frac{1}{\sqrt{s}}\,\frac{1}{\sqrt{(x^2+y^2+z^2)\,s + x^2 + y^2}}$$

and each of these factors is completely monotonic. Finally, by Lemma 1, the above elementary solution is completely monotonic for $0 < s < \infty$. Thus the elementary solution of the modified Stokes' equation (9.1) is positive.

10. Positive Solutions, and a Maximum Principle

A common method of establishing that certain solutions of a partial differential equation are positive is by means of a maximum principle. [3, 4, p. 326; 16]. This method also applies to equations with variable coefficients. Here we will show that the maximum principle technique can be applied to the Laplace transforms of a slightly different class of time-dependent equations in order to show positivity of solutions. In this case we do not work with elementary solutions, but with general regular solutions satisfying suitable positivity conditions.

We consider equations of the form

$$P_0(D)\,U = Q(D)\,L(U) \tag{10.1}$$

where

$$L(U) = \Delta U + b \cdot \nabla u + cu$$

is a general second-order elliptic operator in n dimensions with variable (sufficiently smooth) coefficients which, however, do not depend on t [8, Chapter 4]. We also assume $c \leqslant 0$ as is usually required for a maximum principle. Solutions will be considered in a smooth region R of n-dimensional space, with boundary B; boundary values of u on B at time t will be specified as well as suitable initial values.

THEOREM 5. *Let*

$$P(D)\, U(t, x) = Q(D)\, \Delta U(t, x) + f(t, x), \tag{10.2}$$

where $t \geqslant 0$, $x \in R$, *and* $P(s), Q(s)$ *are polynomials of degrees* $m + 1$, m *respectively, with real, simple, nonpositive zeros in alternating order, and such that*

$$P(s) > 0, \quad Q(s) > 0, \qquad \text{for} \qquad s > 0. \tag{10.3}$$

Let

$$\begin{array}{cccc} U(0, x) \geqslant 0, & U_t(0, x) \geqslant 0, & \cdots & U_t^{(m)}(0, x) \geqslant 0 \\ LU(0, x) \leqslant 0, & LU_t(0, x) \leqslant 0, & \cdots & LU_t^{(m-1)}(0, x) \leqslant 0 \end{array}$$

and $U(t, x) \geqslant 0$ *for* $x \in B$. *Then*

$$U(t, x) \geqslant 0 \qquad x \in R,\ t > 0. \tag{10.4}$$

PROOF. Let

$$\tilde{U}(s, x) = \int_0^\infty e^{-st} U(t, x)\, dt, \tag{10.5}$$

and observe that

$$\widetilde{P(D)\, U(t,x)} = \widetilde{Q(D) LU} + \tilde{f}(s, x) \tag{10.6}$$

where the tilde ~ denotes the Laplace transform of the function indicated in each occurrence.

Since

$$\widetilde{D^k U} = s^k U(s) - \sum_{j=0}^{k-1} s^j U^{(k-j-1)}(0),$$

we find that with

$$P(s) = \sum_{h=0}^{m+1} P_h s^h$$

$$\widetilde{P(D)\, U} = P(s)\, \tilde{U}(s) - \sum_{h=1}^{m+1} P_h \sum_{j=0}^{h-1} s^{h-j-1} U^{(j)}(0)$$

$$= P(s)\, \tilde{U}(s) - \sum_{j=0}^{m} U^{(j)}(0) \sum_{h=1}^{m+1} P_h s^{h-j-1}.$$

The coefficient of $U^{(j)}(0)$ in the latter sum is the Horner polynomial $\Delta_0^{j+1} P(s)$, where

$$\Delta_0 P(s) = \frac{P(s) - P(0)}{s}, \ldots, \Delta_0^{j+1} P(s) = \Delta_0 \Delta_0{}^j P(s),$$

etc., [6, p. 322].

From (10.6) we thus obtain

$$P(s)\, \tilde{U}(s, x) = Q(s)\, L\tilde{U}(s, x) + f(s, x) + \sum_{j=0}^{m} \tilde{U}^{(j)}(0, x)\, \Delta_0^{j+1} P(s)$$

$$- \sum_{j=0}^{m-1} L\tilde{U}^{(j)}(0, x)\, \Delta_0^{j+1} Q(s). \tag{10.7}$$

We now show that $\tilde{U}(s, x)$ is completely monotonic in s for $0 < s < \infty$, i.e., we show by induction on k using a maximum principle that $(-1)^k\, \tilde{U}^{(k)}(s, x) \geqslant 0$. For $k = 0$ note that

$$F(s)\, \tilde{U} = L\tilde{U} + G(s), \tag{10.8}$$

where

$$F(s) = \frac{P(s)}{Q(s)} \tag{10.9}$$

and

$$G(s) = \frac{f(s, x) + \sum_{=0}^{m} \tilde{U}^{j}(0, x)\, \Delta_0^{j+1} P(s) - \sum_{j=0}^{m-1} L\tilde{U}^{(j)}(0, x)\, \Delta_0^{j+1} Q(s)}{Q(s)}. \tag{10.10}$$

Observe that $f(t, x) \geqslant 0$ implies that $f(s, x)$, and hence $f(s, x)/Q(s)$, are completely monotonic. To show that the remaining terms in $G(s)$ are completely monotonic, we require the following lemma.

LEMMA 7. *Let $P(s)$ be a polynomial of degree m or $m + 1$, and $Q(s)$ a polynomial of degree m, with distinct zeros such that*

(a) *all zeros of $P(s)$ and $Q(s)$ are negative (or zero).*

(b) *the zeros of $P(s)$ and $Q(s)$ separate each other.*

(c) *the zero closest to the origin is a zero of $P(s)$.*

(d) *for $s > 0$, $P(s)\, Q(s) > 0$.*

Then

(1) $\dfrac{d}{ds}\left(\dfrac{P(s)}{Q(s)}\right)$

(2) $\dfrac{\Delta_0^{j+1} P(s)}{Q(s)}, \qquad j = 0, 1, 2, \ldots, m-1 \quad \text{or} \quad m,$

(3) $\dfrac{\Delta_0^{j+1} Q(s)}{Q(s)}, \qquad j = 0, 1, 2, \ldots, m-1,$

are completely monotonic for $s > 0$.

PROOF. Consider the case where $P(s)$ has degree $m + 1$, and let

$$P(s) = \prod_{j=1}^{m+1} (s + \alpha_j)$$

$$Q(s) = \prod_{j=1}^{m} (s + \beta_j),$$

where $0 \leqslant \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_m < \alpha_{m+1}$.

Then

$$F(s) = \frac{P(s)}{Q(s)} = \frac{\prod (s + \alpha_j)}{\prod (s + \beta_j)} > 0$$

for $0 < s < \infty$. The partial fraction form of this quotient is

$$F(s) = s + \beta - \sum_{j=1}^{m} \frac{\gamma_j}{s + \beta_j}$$

where

$$\gamma_j = - \lim_{s \to -\beta_j} (s + \beta_j) F(s)$$

$$= - \frac{P(-\beta_j)}{\prod\limits_{\ell \neq j} (\beta_\ell - \beta_j)}.$$

Under the stated conditions the sign of $P(-\beta_j)$ is always opposed to the sign of the product in the denominator, hence

$$\gamma_j > 0 \qquad j = 1, \ldots, m.$$

The first conclusion of the lemma now follows at once, since $F'(s)$ contains only terms that are obviously completely monotonic for $0 < s < \infty$.

To show that the difference expressions in (2) have the desired property we must express $\Delta_0^{j+1} P(s)$ in terms of symmetric functions of the α_j. Thus

$$\Delta_0^{j+1} P(s) = s^{m+1} + P_m s^m + P_{m-1} s^{m-1} + \cdots + P_{j+1}$$

$$= s^{m+1} + \sum \alpha_j s^m + \sum \alpha_j \alpha_k s^{m-1} + \cdots + \sum \alpha_j \cdots \alpha_l ,$$

where the summations indicate elementary symmetric functions of

$\alpha_1, \ldots, \alpha_{m+1}$. We now express $\Delta_0^{j+1}P(s)$ as a sum of factorial type products such such as $(s + \alpha_{m+1})(s + \alpha_m) \cdots (s + \alpha_{j+2})$; in fact it is easily verified that

$$\Delta_0^{j+1}P(s) = \sum_{k=j+1}^{m+1} \sigma_k^{j+1}(s + \alpha_{m+1})(s + \alpha_m) \cdots (s + \alpha_{k+1}),$$

where σ_0^{j+1} is the elementary symmetric function of order $k - j - 1$ of $\alpha_1, \ldots, \alpha_k$, and hence is positive.

Since $\alpha_{m+1} > \beta_m$, $\alpha_m > \beta_{m-1}, \ldots, \alpha_{k+1} > \beta_k$ it follows from Lemma 2 that the product

$$\sigma_k^{j+1} \frac{s + \alpha_{m+1}}{s + \beta_m} \cdot \frac{s + \alpha_m}{s + \beta_{m-1}} \cdots \frac{s + \alpha_{k+1}}{s + \beta_k} \frac{1}{s + \beta_{k-1}} \cdots \frac{1}{s + \beta_1}$$

is completely monotonic, and this yields the result stated for $j = 0, 1, \ldots, m$. The result (3) for differences of $Q(s)$ now follows easily. This completes the proof of Lemma 7.

In particular, the function $G(s)$ defined by (10.10) has now been shown be completely monotonic for $0 < s < \infty$.

Returning to the proof of the theorem, suppose now that $\tilde{U}(s, x)$ takes negative real values in R, and let x_0 be a point where the minimum value of $\tilde{U}(s, x)$ is taken, thus

$$\tilde{U}(s, x_0) \leqslant \tilde{U}(s, x) \qquad x \in R.$$

Since $U(t, x) \geqslant 0$ on B, we have $\tilde{U}(s, x) \geqslant 0$ on B so $x_0 \in R - B$. By the maximum principle, or rather the extremal property $uLu \leqslant 0$ of L, we have $L\tilde{U}(s, x_0) \geqslant 0$. Thus

$$F(s)\, \tilde{U}(s, x_0) < 0$$

while

$$L\tilde{U} + G(s) \geqslant 0$$

so that (10.8) is contradicted and it follows that $\tilde{U}(s, x_0) \geqslant 0$ in R.

Next we show that

$$\tilde{U}^{(1)}(s, x) \equiv \frac{\partial \tilde{U}(s, x)}{\partial s}$$

is nonpositive. Differentiating (10.8) with respect to s, we find

$$F'(s)\, \tilde{U} + F(s)\, \tilde{U}' = L\tilde{U}' + G'(s). \tag{10.11}$$

Supposing that $\tilde{U}'(s, x_0) > 0$, we observe that $x_0 \in R - B$ since $\tilde{U}'(s, x) \leqslant 0$

on B, and that $L\tilde{U}'(s, x_0) \leqslant 0$. The left-side is, since $F(s) > 0$, $F'(s) > 0$, $\tilde{U} > 0$,

$$F'(s)\, \tilde{U} + F(s)\, \tilde{U}'(s, x_0) > 0$$

while the right side is

$$L\tilde{U}' + G'(s) \leqslant 0.$$

The contradiction shows that $\tilde{U}'(s, x) \leqslant 0$ in R.

Proceeding by induction on k, we assume $(-1)^{\ell}\, U^{(\ell)}(s, x) \geqslant 0$ for $\ell = 0, 1, 2,..., k-1$, differentiate (10.8) k times with respect to s and then multiply by $(-1)^k$. Thus

$$\begin{aligned}(-1)^k F^{(k)}(s)\, \tilde{U}(s, x) + (-1)^k k F^{(k-1)}_{(s)}\, \tilde{U}^{(1)}(s, x) + \cdots + F(s)\, (-1)^k\, \tilde{U}^{(k)}(s, x) \\ = (-1)^k L\tilde{U}^{(k)}(s, x) + (-1)^k\, G^{(k)}(s). \qquad (10.12)\end{aligned}$$

Suppose now that $(-1)^k\, \tilde{U}^{(k)}(s, x_0) < 0$. As before we show that $(-1)^k\, \tilde{U}^{(k)}(s, x) \geqslant 0$ on B so that an interior minimum must be present, at which $(-1)^k L\tilde{U}^{(k)}(s, x_0) \geqslant 0$. The left side of (10.12) contains the terms

$$(-1)^k F^{(k)}(s)\, \tilde{U}(s, x) \leqslant 0$$

$$(-1)^k k F^{(k-1)}(s)\, \tilde{U}^{(1)}(s, x) \leqslant 0$$

$$\vdots$$

$$(\quad 1)^k F'(s)\, \tilde{U}^{(k-1)}(s, x) \leqslant 0$$

$$(-1)^k F(s)\, \tilde{U}^{(k)}(s, x) < 0,$$

so that the left side of (10.12) is negative. But both the terms $(-1)^k L\tilde{U}^{(k)}(s, x)$ and $(-1)^k\, G^{(k)}$ on the right side are positive. This contradiction shows that $(-1)^k\, \tilde{U}^{(k)}(s, x) \geqslant 0$ for $s > 0$, $x \in R$ and $k = 0, 1, 2,...$. Thus the Laplace transform of $U(t, x)$ is completely monotonic, and we conclude that

$$U(t, x) \geqslant 0 \qquad t > 0, x \in R$$

as stated. This completes the proof of Theorem 5.

11. A Boundary Condition of the Third Kind

Reasoning of the type used in the preceding section can also be applied to a boundary condition similar to that of the third kind but also containing time derivatives. We consider a region R in n dimensions and a second order elliptic operator

$$L(u) = \Delta u + b \cdot \nabla u + cu, \tag{11.1}$$

where $c \leqslant 0$ so that $L(u)$ satisfies a maximum principle. The coefficients in (11.1) are assumed independent of t. The boundary condition will be

$$P(D)\,u + Q(D)\,\frac{\partial u}{\partial n} = f(P), \tag{11.2}$$

where $P(D)$, $Q(D)$ are polynomials that satisfy the conditions of Theorem 5. For simplicity we shall now restrict consideration to homogeneous initial conditions.

THEOREM 6. *Let $u(t, x)$ satisfy*

$$Lu = f(t, x) \leqslant 0 \qquad t > 0, x \in R, \tag{11.3}$$

and the boundary condition of the third kind

$$P(D)\,u + Q(D)\,\frac{\partial u}{\partial n} = g(t, x) \geqslant 0, \qquad t > 0, x \in B, \tag{11.4}$$

and let $u(t, x)$ together with its first m time derivatives vanish as $t \to 0$. Then

$$u(t, x) \geqslant 0, \qquad t > 0, x \in R. \tag{11.5}$$

PROOF. As before, let $\tilde{U}(s, x)$ denote the Laplace transform of $u(t, x)$. Then

$$L\tilde{U}(s, x) = \tilde{f}(s, x) \tag{11.6}$$

and $\tilde{f}(s, x)$ is completely monotonic for $s > 0$. Thus

$$L((-1)^k\,\tilde{U}^{(k)}(s, x)) \geqslant 0 \qquad k = 0, 1, 2, \ldots \tag{11.7}$$

so that $(-1)^k\,\tilde{U}^{(k)}(s, x)$ is "superharmonic" and has no negative minimum in the interior of R.

Transforming the boundary condition, we find, because of the homogeneous initial conditions, that

$$P(s)\,\tilde{U}(s, x) + Q(s)\,\frac{\partial \tilde{U}(s, x)}{\partial n} = \tilde{g}(s, x), \tag{11.8}$$

the right-hand side being completely monotonic for $0 < s < \infty$. Setting

$$F(s) = \frac{P(s)}{Q(s)}, \tag{11.9}$$

we have after division by $Q(s)$,

$$F(s)\,\tilde{U}(s, x) + \frac{\partial \tilde{U}(s, x)}{\partial n} = G(s, x), \tag{11.10}$$

where $G(s, x)$ and $F'(s)$ are completely monotonic for $s > 0$.

Suppose now that $\tilde{U}(s, x)$ is somewhere negative in $R + B$. Then $\tilde{U}(s, x)$ has a negative minimum, say at x_0; by (11.7) x_0 lies on the boundary of R. With n denoting the outward normal to the boundary, we have

$$\frac{\partial \tilde{U}(s, x_0)}{\partial n} \leqslant 0 \qquad x_0 \in B. \tag{11.11}$$

But $F(s) > 0$ so

$$F(s)\,U(s, x) + \frac{\partial \tilde{U}(s, x_0)}{\partial n} < 0, \tag{11.12}$$

while $G(s, x_0) \geqslant 0$, which contradicts (11.10). Thus $\tilde{U}(s, x)$ must be non-negative in $R + B$.

Proceeding by induction on the order of derivatives, we differentiate (11.10) k times with respect to s and multiply by $(-1)^k$. Thus

$$\begin{aligned}(-1)^k F^{(k)}(s)\,\tilde{U}(s,x) + (-1)^k F^{(k-1)}(s)\,\tilde{U}'(s,x) + \cdots + F'(s)\,(-1)^k \tilde{U}^{(k-1)}(s,x)\\ + F(s)\,(-1)^k\,\tilde{U}^{(k)}(s, x) + (-1)^k \frac{\partial \tilde{U}^{(k)}(s, x)}{\partial n}\\ = (-1)^k\,G^{(k)}(s, x) \geqslant 0.\end{aligned} \tag{11.13}$$

As induction hypothesis we suppose

$$(-1)^{\ell}\,\tilde{U}^{(\ell)}(s, x) \geqslant 0, \qquad \ell = 0, 1, \ldots, k-1. \tag{11.14}$$

Suppose that $(-1)^k\,\tilde{U}^{(k)}(s, x)$ is somewhere negative in $R + B$. By (11.7) this function has any negative minimum value at the boundary-say at $x_0 \in B$. Then

$$(-1)^k \frac{\partial \tilde{U}^{(k)}}{\partial n}(s, x_0) \leqslant 0.$$

Since $F'(s)$ is completely monotonic by hypothesis, we have

$$(-1)^k F^{(k)}(s) \leqslant 0 \qquad k = 0, 1, 2, \ldots .$$

It now follows that every term on the left of (11.13) is nonpositive, while the term $F(s)\,(-1)^k\,\tilde{U}^{(k)}(s, x)$ is actually negative. This contradicts (11.13) and shows that

$$(-1)^k\,\tilde{U}^{(k)}(s, x) \geqslant 0 \qquad s > 0, x \in R + B,$$

holds for k; by induction it also holds for all positive integer values of k. Invoking Bernstein's theorem we now deduce (11.5). This completes the proof of Theorem 6.

Combining the methods used in the proofs of Theorems 5 and 6, we can establish the following further result.

THEOREM 7. *Let $P(D)$, $Q(D)$, and $p(D)$, $q(D)$ be two pairs of polynomials each pair of which satisfies the conditions of Theorem 5. If $u(t, x)$ vanishes to a sufficiently high order as $t \to 0^+$, and if for $t > 0$*

$$P(D)\, u - Q(D)\, \Delta u \geqslant 0 \qquad \text{in} \qquad R,$$

while

$$p(D)\, u + q(D) \frac{\partial u}{\partial n} \geqslant 0 \qquad \text{on} \qquad B = \partial R,$$

then

$$u(t, x) \geqslant 0 \qquad \text{in} \qquad R, t > 0.$$

Details of the proof are left to the reader. At each stage of a similar inductive proof one shows that negative values are impossible, since a negative minimum cannot occur either within the region, or on the boundary.

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